A note on the relationships between maximal informativity and exhaustivity

Keny Chatain (MIT) - kchatain@mit.edu

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[INCOMPLETE DRAFT - comments are welcome]

1 The case of questions

There are mutual connections between exhaustivity and maximal informativity. If a rational speaker knows that the proposition expressed by (1)b is maximally informative with respect to the question (1)a, given the current state of affairs, she can infer the exhaustified statement (1)c.

(1) a. Who jazzed?
 b. Electra and Antigone jazzed.
 c. Electra and Antigone and no one else jazzed.

So the link between maximal informativity and exhaustivity is the following: the proposition expressing "*p is maximally informative with respect to Q*" is the proposition obtained by exhaustifying p with respect to Q

(2) $Exh(p)(Q) = \lambda w.p$ is maximally informative in Q given $w = \lambda w.p \in$ **q-max-inf**$_w(Q)$

We can further simplify this equality even further by treating the subscript of **q-max-inf** as a normal argument of that operator and equating sets with characteristic functions.

(3) $Exh(p)(Q)(w) =$ **q-max-inf**$(w)(Q)(p)$

So going from exhaustivity to maximal informativity is just a matter of swapping arguments. We think of Exh as returning a proposition - a set of worlds - on input of a proposition (the prejacent) and a set of propositions (the alternatives). So the type of this function is $(st)((st)t)st$. And we think of maximal-informativity as taking a set of propositions (the question) and a world (the current state of affairs) and returning a set of propositions (the maximal informative propositions in the current set of affairs). So the type of this mapping is $s((st)t)(st)t$. Although conceptualized differently, these operator are identical, up to argument reordering.

Theoretically, this correspondence means that to any flavour of exhaustivity will correspond a different flavour of maximal informativity. I list in the chart below which notion of exhaustivity correspond to which notion of maximal informativity:

(4)

Exhaustivity	**Maximal Informativity**
Negate all stronger alternatives	Greatest[1] element for entailment among true answers
Negate all non-weaker alternatives	Maximal[1] element for entailment among true answers
Negate innocently excludable alternatives	?

2 The case of intensional properties

Another notion of maximal informativity is used when dealing with properties. I know of two linguistic examples: maximal informativity for degree predicates (Beck, 2014) and maximal informativity for definite descriptions (Iatridou et al., 2014). The procedure is the same in both cases ; some operator applies to a predicate and returns the set of all those elements that are maximally informative in the predicate. Here are two definitions for each of these cases:

(5) a. $[\![\textbf{deg-max-inf}]\!] = \lambda D_{dst}.\lambda d.\lambda w.\ D(d)(w) \wedge \forall d', (D(d')(w)) \rightarrow \big(D(d) \Rightarrow D(d')\big)$

b. $[\![\textbf{def-max-inf}]\!] = \lambda P_{est}.\lambda x.\lambda w.\quad P(x)(w) \wedge \forall x', (P(x')(w)) \rightarrow \big(P(x) \Rightarrow P(x')\big)$

One can be more general and define for any type a, what it means for an object of type a to be the maximally informative element of an intensional property given a state of affairs w. This generalized maximal informativity operator will take an intensional property P_{ast}, an object of type a and a state of affairs w and returns true if the object is maximally informative in P given state of affairs w. So the type of this mapping is $a(ast)st$.

(6) $[\![\textbf{max-inf}_a]\!] = \lambda P_{ast}.\lambda x_a.\lambda w.P(x)(w) \wedge \forall x', (P(x')(w)) \rightarrow \big(P(x) \Rightarrow P(x')\big)$

This general definition covers the particular examples of Iatridou et al. (2014) and Beck (2014): **deg-max-inf** is just $\textbf{max-inf}_d$, **def-max-inf** is just $\textbf{max-inf}_e$.

Intriguingly, this definition does not subsume the notion of maximal informativity defined for questions, as we saw from the previous section. Contrary to what we might expect, **q-max-inf** is not $\textbf{max-inf}_{st}$, the operator of maximal informativity for

[1] A greatest element among a set is an element that no element of the set is greater than ; a maximal element is an element that is greater than all other elements

propositions. Whereas **q-max-inf** is a mapping of type $(st)\underline{((st)t)}st$, $\mathbf{max\text{-}inf}_{st}$ is of type $(st)((st)st)st$.

To identify the two notions, one would need the question argument of **q-max-inf** (underlined) to be a property of propositions (type $(st)st$), not a set of proposition (type st). We can have this. Instead of defining the denotation of questions as a mere set of propositions, we define questions as functions from worlds to the set of answers that are true in that world. The mapping from the old notion to the new notion is formally represented below:

(7) $Q \quad \rightsquigarrow \quad Q' = \lambda p.\lambda w.p \in Q \wedge p(w)$

In the appendix 3.2, I give a linguistic reason why the semantics of question already force this upon us. With this shift, the two notions of maximal informativity are reconciled:

(8) For all question Q (and its new shifted definition Q'), p and w,
$\mathbf{q\text{-}max\text{-}inf}(w)(Q)(p) = \mathbf{max\text{-}inf}_{st}(Q')(w)(p)$

So the notion of maximal informativity used in questions and in properties are the same. But can we get parallel notions of exhaustivity for properties? The next section tries to answer that question in the general case.

3 Exhaustivity and maximal informativity: the general case

3.1 Meta-language operators.

As we saw in the first section, **q-max-inf** was straightforwardly linked to Exh. But what connects to Exh to $\mathbf{max\text{-}inf}_a$? This question is crucial because as we saw, each new definition of Exh brings about a new notion of maximal informativity. Insights gotten from the former notion can feed analysis of the latter.

Taking the hint from the relationship between **q-max-inf** and Exh, one could define a notion of Exh' from $\mathbf{max\text{-}inf}_a$, simply by swapping around the arguments that the arguments that $\mathbf{max\text{-}inf}_a$ takes:

(9) $Exh(x)(P) = \lambda w.\mathbf{max\text{-}inf}_a(w)(P)(x)$

We thus define a notion of exhaustification that exhaustifies objects against a set of alternative objects. This isn't quite remote from the notion of exhaustivity that we are used to, which is based on propositions (or at least t-ending types). However, recall that **q-max-inf** is straightforwardly connected to Exh and that **q-max-inf** is itself definable in terms of $\mathbf{max\text{-}inf}_{st}$. Piecing these conclusions together, we can define Exh from $\mathbf{max\text{-}inf}_{st}$ in the following manner:

(10) $Exh(p)(Q) = \lambda w.\mathbf{max\text{-}inf}_{\mathrm{st}}(w)(Q')(x)$
where Q' is the lift of Q as proposed in (7)

We are more interested in the other route: how could we define maximal informativity in terms of an exhaustification operator? Since Exh operates on the basis of a proposition and a set of alternative propositions, we must find a way to convert from objects of type a and properties of such objects to propositions.

I propose the following conversion: to each object x in the domain D_a, corresponds the proposition $P(x)$ "*λw. the object x belongs to P in w*" ; the alternatives are the set of all such propositions, for all values of x:

(11) $\mathbf{max\text{-}inf}_{\mathrm{a}} = \lambda P_{ast}.\lambda x_a.Exh(P(x))(\bigcup P)$
where $\bigcup P = \{P(y) \mid y \in D_a\}$

This proposal is formally rendered in (11). We just need to check that it fits the bill: for sensible definitions of Exh, does this define the notions of maximal informativity that we are used to? The answer is yes. Take for instance Exh to negate all the non-weaker alternatives to its prejacent ; then $\mathbf{max\text{-}inf}_a$ reduces to:

(12)

$$\begin{aligned}\mathbf{max\text{-}inf}_{\mathrm{a}} &= \lambda P_{ast}.\lambda x_a.Exh(P(x))\left(\bigcup P\right)\\ &= \lambda P_{ast}.\lambda x_a.\lambda w.P(x)(w) \wedge \forall y \in D_a, (P(y) \not\Rightarrow P(x)) \rightarrow \neg P(y)(w)\\ &= \lambda P_{ast}.\lambda x_a.\lambda w.P(x)(w) \wedge \forall y \in D_a, P(y)(w) \rightarrow (P(x) \Rightarrow P(y))\end{aligned}$$

This is the definition for $\mathbf{max\text{-}inf}_a$ we started with. But this is just one of the many possibilities: when our definition of Exh changes, so will our notion of maximal informativity. As an interesting application, let's consider what would happen if $\mathbf{max\text{-}inf}_a$ were defined in terms of innocent exclusion:

(13)

$$\begin{aligned}\mathbf{max\text{-}inf}_{\mathrm{a}IE} &= \lambda P_{ast}.\lambda x_a.\underset{\mathrm{IE}}{Exh}(P(x))\left(\bigcup P\right)\\ &= \lambda P_{ast}.\lambda x_a.\lambda w.P(x)(w) \wedge \forall y \in D_a, P(y) \in \mathrm{IE}_w\left(P(x), \bigcup P\right) \rightarrow \neg P(y)(w)\end{aligned}$$

Often, $\mathbf{max\text{-}inf}_a$ operators are used in conjunction with existence and uniqueness presuppositions (Iatridou et al., 2014) as in (14). A natural question to ask is this: is the presupposition of existence and uniqueness of a maximally informative element satisfied in the same set of worlds if we switch to $\mathbf{max\text{-}inf}_{a\mathrm{IE}}$?

(14) **E&U presupposition:** In world w, there exists a unique object x such that $\mathbf{max\text{-}inf}_{\mathrm{a}}(P)(x)$ is true in w

Before we tackle that question, note that for instance, maximal informativity defined in terms of greatest element for entailment and maximal informativity defined in terms of maximal element (see 4) trigger the same existence and uniqueness presupposition. This is because existence and uniqueness of a greatest element in a set is equivalent to existence and uniqueness of a maximal element[2].

With innocent exclusion, the E&U presupposition will be different. For instance, consider the highly artificial predicate "λx. *some atom of x is smiling*" and the a domain with two individuals *Joana* and *Marius*, and one plurality *Joana* ⊕ Marius. Let j and m be the propositions "*Joana is smiling*" or *Marius is smiling.* Then we have the following equivalences:

(15) a. $P(\text{Joana}) = j$

b. $P(\text{Marius}) = m$

c. $P(\text{Joana} \oplus \text{Marius}) = j \vee m$

Our translation key between Exh and **max-inf**$_s$tates that the words in which an object x is maximally informative are the worlds in which exhaustification of $P(x)$ against the alternatives of the form $P(y)$ is true. Applying this to our examples, we get:

(16) IE Exhaust

a. Worlds where Joana is maximally informative: $j \wedge \neg m$

b. Worlds where Marius is maximally informative: $m \wedge \neg j$

c. Worlds where Joana⊕Marius is maximally informative: $m \vee j$

So we can see that the only worlds where there is a unique maximally informative object are the worlds in which both m and j are true. Also note that counterintuitively, Joana⊕Marius makes the weakest statement but is always maximally informative. Compare this result with the result obtained with another Exh operator in (17). Here the E&U presupposition is satisfied if only one of j and m is true.

(17) Negate non-weaker alternatives

a. Worlds where Joana is maximally informative: $j \wedge \neg m$

b. Worlds where Marius is maximally informative: $m \wedge \neg j$

c. Worlds where Joana⊕Marius is maximally informative: $\emptyset$

The E&U presuppositions of IE exhaustification and non-weaker exhaustification are different. In the linguistic cases where one needs **max-inf**$_a$, which usually involves a E&U presupposition, we might then expect to see a difference.

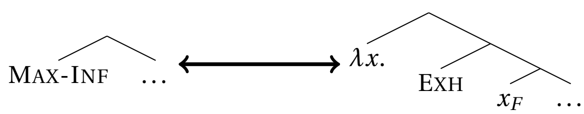
3.2 Linguistic operators

Our discussion has so far focused on meta-language operators, but these are eventually instantiated by covert object language operators. This prompts a question:

[2] In finite domains...

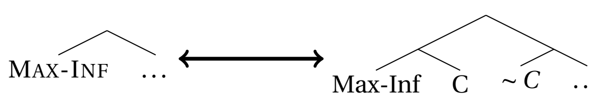
given that exhaustivity and maximal-informativity are interdefinable, can we rid our inventory of covert operators for the object language from one of the two notions and how? Can we have Exh play in the LF the role once assigned to Max-Inf and vice-versa? The answer is yes, but always at a cost.

Defining Max-Inf in terms of Exh. Starting one way, suppose an operator Max-Inf is applied to a node *ast* in a tree. Our previous discussion informs us that this LF will be equivalent to one where a dummy variable is inserted for the purpose of exhaustification. Importantly, that variable and that variable alone must be used to form alternatives (which I indicate with the subscript F)

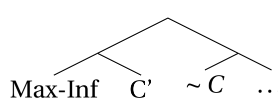
(18)


To be entirely equivalent, the alternatives formed by replacing x in the LF of (18) must all possible other value of the variable (i.e. no pruning happens). All in all, this translation incurs two costs: inserting dummy abstraction and rigid alternatives.

Defining Exh in terms of Max-Inf The other way around, can we replace Exh with Max-Inf? This turns to be similarly tricky. Max-Inf requires a predicate of objects as its first argument. Here the relevant objects are propositions. This argument could be provided in the form of a covert variable constrained by focus, as such:

(19)


However, the alternatives generated by focus are not of the right type: they are $(st)t$, when Max-Inf demands predicate of the form $(st)st$. This problem, already solved above is solved if we use the type shifter $Q \rightsquigarrow Q'$, already seen earlier. The final LF looks as follows:

(20)


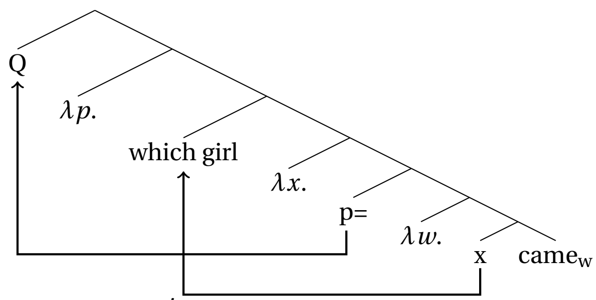
In a nutshell, the main costs of this reduction are: 1) assuming that the predicate argument of Max-Inf is covert and constrained by focus, 2) the type-shifter on the set of alternatives.

Appendix: question intension

I proposed to replace questions as set of propositions (type $(st)t$) with questions as properties to propositions (type $(st)st$). This unorthodox representation turns out to be already implicit in our semantics. Take a standard Hamblin question semantics. *wh*-words are regular existentials that scope above a proposition forming operator:

(21) a. which girl came?

b.



c. $[\![(21b)]\!] = \lambda p.\, \exists x \in \texttt{girl}',\ p = \lambda w.\, \text{x came}_\text{w}$

This standard derivation however is problematic. On the one hand, I take intensionality seriously by indexing the predicate *came* to the world it is bound to. On the other hand, I allow myself to not specify which world the predicate *girl* is bound to. To fix this, one may want to use w_0, the "*actual world*".

(22) $[\![(21b)]\!]^{w_0} = \lambda p.\, \exists x \in \texttt{girl}'_{w_0},\ p = \lambda w.\, \text{x came}_\text{w}$

But this only makes the problem more blatant. In the dynamics of conversation, we don't know what the actual world is ; statements give information about how the actual world is like. This narrows down the set of possibilities that we entertain for the actual world (a.k.a. the context set).

Questions, however, are ways of signaling what information is deemed relevant and do not say anything about the actual world. As such, their denotation should not depend on what the actual world is like. But this is exactly what the denotation in (26b) seems to yield: the set of answers depends on the set of girls in the actual world!

What if we required explicitly of questions that they do not depend on the actual world ? In other words, that their intensions be constant on the context set. For the denotation in (22), that would mean that the set of girls is the same in all worlds of the context set. In turn, that implies that it is common ground who all the girls are. This is too strong a requirement ; as (23), a *which*-question like (23) does not require me to know who is Pole and who isn't.

(23) Which Pole won the Nobel prize?

But the paradox is only apparent. Biting the bullet, let's accept the world-dependence of Hamblin sets. To explicitly encode the intensionality of questions, we can use world variables and binding and represent the Hamblin set as follows:

(24) $[\![(21b)]\!] = \lambda p.\lambda w'.\ \exists x \in \texttt{girl}'_{w'},\ p = \lambda w.\ \text{x came}_{\text{w}}$

The paradox is resolved if we notice that the Hamblin set of a question can be world-dependent while the partition induced by it need not be. We just need to find a procedure that turns a world-dependent Hamblin set into a partition of worlds that is not world-dependent. Here, we can use a standard idea: w_1 and w_2 are in the same cell of the question partition iff they have the same set of true answers. This is formalized in (25).

(25) $w_1 \sim_Q w_2$ iff $\forall p, Q(p)(w_1) \wedge p(w_1) \leftrightarrow Q(p)(w_2) \wedge p(w_2)$

As we can see, the relation of $\sim_Q$ is defined without making reference to the actual world, so we ensure that the relevance partition of the common ground is world-independent.

If one accepts the picture painted here, one is committed to questions having something like the type in (24), namely $(st)st$. This is the type obtained by the type-shifter $Q \rightsquigarrow Q'$. So the type-shifter turns out to be dispensable. There is a final problem: even though the type of questions is now right, we don't have quite the result obtained by the type shifter: whereas the type-shifter gave in a world the set of answers *true* in that world, our current denotation yields the set of all answers in that world.

We can fix this by tweaking the meaning of the Q operator in (21b) so that it throws out false propositions from the set of answers as in (26)a ; with this modified denotation, the denotation of (21b) is as in (26)b. This is exactly the question denotation we need to reconcile **q-max-inf** and **max-inf**$_{\text{a}}$.

(26) a. $[\![\text{Q}]\!] = \lambda \text{ANS}_{(st)st}.\lambda p_{st}.\lambda w.p \in \text{ANS} \wedge p(w)$

b. $[\![(21b)]\!] = \lambda p.\lambda w'.\ \exists x \in \texttt{girl}'_{w'},\ p = \lambda w.\ \text{x came}_{\text{w}} \wedge x \text{ came}_{w'}$

References

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